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Super-Yangian $Y(gl(1|1))$ and its oscillator realization

Guo-xing Ju†‡, Jin-fang Cai†, Han-ying Guo†, Ke Wu† and Shi-kun Wang§

† Institute of Theoretical Physics, Academia Sinica, Beijing 100080, People's Republic of China‖
‡ Physics Department, Henan Normal University, Xinxiang, Henan Province 453002, People's Republic of China
§ CCAST(World Laboratory), Beijing 100080, People's Republic of China and Institute of Applied Mathematics, Academia Sinica, Beijing 100080, People's Republic of China

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Abstract. On the basis of graded RTT formalism, the defining relation of the super-Yangian $Y(gl(1|1))$ is derived and its oscillator realization is constructed.

Yangian $Y(g)$ of a simple Lie algebra g, first introduced by Drinfeld [1], is a deformation of the universal enveloping algebra $U(g[t])$ of a current algebra $g[t]$. It is a kind of Hopf algebra and the tensor products of its finite-dimensional representations produce rational solutions of the quantum Yang–Baxter equation (QYBE).

In the last decade, Yangians associated with simple Lie algebras have been systematically studied both in mathematics and physics [2], and have many applications in such theoretical physics as quantum field theory and statistical mechanics. Yangian structure is the underlying symmetry of many types of integrable models. For example, the one-dimensional Hubbard model on the infinite chain [3], the Haldane–Shastry model [4] and the Polychronakos–Frahm model [5] have Yangian symmetry; in the massive two-dimensional quantum field theory, an infinite-dimensional symmetry generated by non-local conserved currents is connected to the Yangian [6].

As generalizations of Yangians of simple Lie algebras, the Yangians associated with the simple Lie superalgebra, which we will call super-Yangian in this paper, also need to be studied. Actually, some structural features of super-Yangian were investigated by Nazarov [7] and Zhang [8, 9]. In [7], the quantum determinant of the super-Yangian $Y(gl(m|n))$ is described, while in [8, 9] the super-Yangian $Y_q(gl(m|n))$ associated with the Perk–Schultz $\mathcal{R}$ matrix is constructed, its structural properties and the relationship between its central elements and the Casimir operators of quantum supergroup $U_q(gl(m|n))$ are discussed, in particular, the classification of the finite-dimensional irreducible representations of the super-Yangian $Y(gl(1|1))$ and $Y(gl(m|n))$ is given.

In this paper, on the basis of the graded RTT formalism, we derive the defining relations of the super-Yangian for the Lie superalgebra $gl(1|1)$ and give its oscillator realization. First, we briefly review the graded RTT formalism and the corresponding graded Yang–Baxter equation (GYBE). Then, we give the algebraic relation that super-Yangian $Y(gl(1|1))$ satisfies and construct its oscillator realization. Finally, we make some remarks and discussions.

‖ Mailing address.

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In the supersymmetric case, space is graded and the tensor product has the following property

$$(A \otimes B)(C \otimes D) = (-1)^{p(B)p(C)} AC \otimes BD \tag{1}$$

where $p(A)$ denotes the degree of A. Now the graded RTT relation with the spectral parameters takes the form [10, 11]

$$\mathcal{R}_{12}(u-v)T_1(u)\eta_{12}T_2(v)\eta_{12} = \eta_{12}T_2(v)\eta_{12}T_1(u)\mathcal{R}_{12}(u-v) \tag{2a}$$

where $T_1(u) = T(u) \otimes 1$ and $T_2(u) = 1 \otimes T(u)$ and $(\eta_{12})_{ab,cd} = (-1)^{p(a)p(b)}\delta_{ac}\delta_{bd}$ and GYBE with spectral parameters reads as [10, 11]

$$\eta_{12}\mathcal{R}_{12}(u)\eta_{13}\mathcal{R}_{13}(u+v)\eta_{23}\mathcal{R}_{23}(v) = \eta_{23}\mathcal{R}_{23}(v)\eta_{13}\mathcal{R}_{13}(u+v)\eta_{12}\mathcal{R}_{12}(u). \tag{3a}$$

Considering the charge conservation conditions for the $\mathcal{R}_{ab,cd}$, i.e.

$$\mathcal{R}_{ab,cd} = 0 \qquad \text{unless } a+b = c+d \tag{4}$$

we can write equations (2a) and (3a) in the component forms as follows

$$(-1)^{p(e)(p(d)+p(f))}\mathcal{R}_{12}(u-v)_{ab,cd}T(u)_{ce}T(v)_{df}$$
$$= (-1)^{p(a)(p(d)+p(b))}T(v)_{be}T(u)_{ad}\mathcal{R}_{12}(u-v)_{cd,ef} \tag{2b}$$
$$(-1)^{p(d)(p(b)+p(e))}\mathcal{R}(u)_{ab,cd}\mathcal{R}(u+v)_{ce,fh}\mathcal{R}(v)_{dh,ij}$$
$$= (-1)^{p(d)(p(h)+p(j))}\mathcal{R}(v)_{be,dh}\mathcal{R}(u+v)_{ah,cj}\mathcal{R}(u)_{cd,fi} \tag{3b}$$

where the repeated indices are understood to take summation. Note that, in equations (2) and (3) the grading property is taken into account by introducing the factor η_{12}. If we set $\eta = 1$, then equations (2) and (3) reduce to the usual RTT relation and YBE respectively.

It is well known that

$$\mathcal{R}_{12}(u) = u + \mathcal{P}_{12} \tag{5}$$

satisfies GYBE (3), where

$$\mathcal{P}_{12} = \eta_{12}P_{12} \tag{6}$$

P stands for the usual permutation operator, i.e. $P(u \otimes v) = v \otimes u$. Substituting equation (5) into equation (2) and introducing the notation

$$[T(u)_{ab}, T(v)_{cd}\} = T(u)_{ab}T(v)_{cd} - (-1)^{(p(a)+p(b))(p(c)+p(d))}T(v)_{cd}T(u)_{ab} \tag{7}$$

we obtain the following relations:

$$(u-v)[T(u)_{ab}, T(v)_{cd}\} + (-1)^{p(a)p(c)+p(a)p(b)+p(b)p(c)}(T(u)_{cb}T(v)_{ad} - T(v)_{cb}T(u)_{ad})$$
$$= 0. \tag{8}$$

Let

$$T(u)_{ab} = \sum_{n=0}^{\infty} u^{-n}T_{ab}^{(n)} \tag{9}$$

then from equation (8), we have

$$[T_{ab}^{(0)}, T_{cd}^{(n)}\} = 0 \tag{10}$$

$$[T_{ab}^{(n+1)}, T_{cd}^{(m)}\} - [T_{ab}^{(n)}, T_{cd}^{(m+1)}\} + (-1)^{p(a)p(c)+p(a)p(b)+p(b)p(c)}(T_{cb}^{(n)}T_{ad}^{(m)} - T_{cb}^{(m)}T_{ad}^{(n)}) = 0. \tag{11a}$$

Similar to the discussion for the Yangian [2], equation (11a) can be rewritten in the following equivalent form:

$$[T_{ab}^{(n)}, T_{cd}^{(m)}\} = (-1)^{1+p(a)p(c)+p(a)p(b)+p(b)p(c)} \sum_{i=0}^{\min(n,m)-1} (T_{cb}^{(i)} T_{ad}^{(m+n-i-1)} - T_{cb}^{(m+n-i-1)} T_{ad}^{(i)}). \tag{11b}$$

In particular, for the case of $a = c$, $b = d$ in the above equation, we have

$$[T_{ab}^{(n)}, T_{ab}^{(m)}\} = (-1)^{1+p(a)p(a)} \sum_{i=0}^{\min(n,m)-1} [T_{ab}^{(i)}, T_{ab}^{(m+n-i-1)}] \tag{12}$$

this shows that $T_{ab}^{(n)}$, with $a \neq b$ and different n $(n > 1)$ will neither commute nor anticommute.

From equation (11b), we know that the following property holds

$$[T_{ab}^{(n)}, T_{cd}^{(m)}\} = [T_{ab}^{(m)}, T_{cd}^{(n)}\} \qquad (n, m \geqslant 1). \tag{13}$$

Here we note that in the non-graded case, equation (13) will give the relation

$$[T_{ab}^{(n)}, T_{cd}^{(m)}] = 0 \qquad (n, m \geqslant 1). \tag{14}$$

For the case of superalgebra $gl(1|1)$, $a = 1, 2$ and $\mathcal{P}$ takes the form

$$\mathcal{P}_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$T(u)$ is a 2×2 matrix. Because of relation (10), we can choose $T^{(0)}$ to be of the form

$$T^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{15}$$

up to a constant factor. Here we should stress that (15) is only a choice, which is different from the non-graded case in that there it is the result of Schur's lemma [12]. From equation (9), we see that equation (15) is equivalent to imposing the asymptotic condition $T(u) \to 1$ for $u \to \infty$. With equations (10), (11) and (15), we obtain the following relations:

$$\begin{cases} [T_3^{(n)}, T_{12}^{(1)}] = [T_3^{(1)}, T_{12}^{(n)}] = 0 \\ [T_3^{(n)}, T_{21}^{(1)}] = [T_3^{(1)}, T_{21}^{(n)}] = 0 \\ [T_0^{(n)}, T_{12}^{(1)}] = [T_0^{(1)}, T_{12}^{(n)}] = -2T_{12}^{(n)} \qquad \text{(for any } n) \\ [T_0^{(n)}, T_{21}^{(1)}] = [T_0^{(1)}, T_{21}^{(n)}] = 2T_{21}^{(n)} \\ \{T_{12}^{(n)}, T_{21}^{(1)}\} = -T_3^{(n)} \end{cases} \tag{16}$$

$$\begin{cases} [T_0^{(2)}, T_3^{(2)}] + 2(T_{21}^{(1)} T_{12}^{(2)} - T_{21}^{(2)} T_{12}^{(1)}) = 0 \\ [T_3^{(n)}, T_{12}^{(2)}] - T_{12}^{(1)} T_3^{(n)} + T_{12}^{(n)} T_3^{(1)} = 0 \qquad (n \geqslant 1) \\ [T_3^{(n)}, T_{21}^{(2)}] + T_{21}^{(1)} T_3^{(n)} - T_{21}^{(n)} T_3^{(1)} = 0 \qquad (n \geqslant 1) \end{cases} \tag{17}$$

and

$$\begin{cases} -T_{12}^{(n+1)} = (2)^{-1}\{[T_0^{(n)}, T_{12}^{(2)}] + T_{12}^{(n)} T_0^{(1)} - T_{12}^{(1)} T_0^{(n)}\} \\ T_{21}^{(n+1)} = (2)^{-1}\{[T_0^{(n)}, T_{21}^{(2)}] + T_{21}^{(1)} T_0^{(n)} - T_{21}^{(n)} T_0^{(1)}\} \\ T_3^{(n+1)} = -\{T_{12}^{(n)}, T_{21}^{(2)}\} + T_{22}^{(1)} T_{11}^{(n)} - T_{22}^{(n)} T_{11}^{(1)} \qquad (n \geqslant 2) \end{cases} \tag{18}$$

where

$$T_3^{(n)} = T_{22}^{(n)} - T_{11}^{(n)} \qquad T_0^{(n)} = T_{22}^{(n)} + T_{11}^{(n)}. \tag{19}$$

From the iterative relation (18), we see that only $T_{ab}^{(1)}$, $T_{ab}^{(2)}$ are basic operators. Now, if we make the following correspondence

$$\begin{cases} T_3^{(1)} = -\gamma_0 z_0 & T_3^{(2)} = -\gamma_1 z_1 \\ T_{12}^{(1)} = \alpha_0 e_0 & T_{12}^{(2)} = \alpha_1 e_1 \\ T_{21}^{(1)} = \beta_0 f_0 & T_{21}^{(2)} = \beta_1 f_1 \\ T_0^{(1)} = -2h_0 & T_0^{(2)} = \delta h_1 \end{cases} \tag{20}$$

and take the choice

$$\alpha_0\beta_0 = \gamma_0 \qquad \alpha_0\beta_1 = \alpha_1\beta_0 = \gamma_1 \qquad \alpha_0\delta = -2\alpha_1 \tag{21}$$

then from equations (16)–(18) we obtain following algebraic relations

$$\begin{cases} e_0^2 = f_0^2 = 0, [h_0, e_0] = e_0 & [h_0, f_0] = -f_0 \\ [z_0, e_0] = [z_0, f_0] = [h_0, z_0] = 0 & \{e_0, f_0\} = z_0 \end{cases} \tag{22}$$

and

$$\begin{cases} [z_1, e_0] = [z_1, f_0] = [z_1, z_0] = [z_1, h_0] = 0 & \\ [f_1, z_0] = 0 & [f_1, h_0] = f_1 \\ \{f_1, e_0\} = z_1 & \{f_1, f_0\} = 0 \\ \{e_1, e_0\} = 0 & \{e_1, f_0\} = z_1 \\ [e_1, z_0] = 0 & [e_1, h_0] = -e_1 \\ [h_1, z_0] = [h_1, h_0] = 0 & \\ [h_1, e_0] = e_1 & [h_1, f_0] = -f_1. \end{cases} \tag{23}$$

Equation (22) is just the defining relation of the Lie superalgebra $gl(1|1)$. Taking the correspondence

$$z_0 \longrightarrow N + M \qquad e_0 \longrightarrow x \qquad f_0 \longrightarrow y \qquad h_0 \longrightarrow N \tag{24}$$

equation (22) will give the same result as that of Liao and Song [11] in the limit $q \longrightarrow 1$. Equation (23) shows that e_1, f_1, h_1, z_1 form a representation of equation (22). e_i, f_i, h_i, z_i $(i = 0, 1)$ also satisfy Serre relations:

$$\begin{aligned} &[z_1, \{e_1, f_1\}] = C_0 z_0 (f_0 e_1 - f_1 e_0) \\ &2\{e_1, [h_1, e_1]\} + C_0[e_0, e_1] + 2C_1[h_1, e_1]e_0 = 0 \\ &2\{f_1, [h_1, f_1]\} + C_0[f_0, f_1] + 2C_1[h_1, f_1]f_0 = 0 \\ &\{e_1, [z_1, e_1]\} + C_0 e_1 e_0 z_0 = 0 \\ &\{f_1, [z_1, f_1]\} + C_0 f_1 f_0 z_0 = 0 \\ &[e_1, \{e_1, f_1\}] + [z_1, [h_1, e_1]] = C_1([e_1, h_1]z_0 + z_1 e_1) + C_0 e_0 (f_0 e_1 - f_1 e_0) \\ &[f_1, \{e_1, f_1\}] - [z_1, [h_1, f_1]] = C_1([f_1, h_1]z_0 - z_1 f_1) + C_0 f_0 (f_0 e_1 - f_1 e_0) \\ &[h_1, \{e_1, f_1\}] - C_1(f_0[h_1, e_1] + [h_1, f_1]e_0) - C_0(f_0 e_1 - f_1 e_0) = 0 \end{aligned} \tag{25}$$

where

$$C_0 = \gamma_0/\alpha_1\beta_1 \qquad C_1 = \gamma_1/\alpha_1\beta_1.$$

The operators $\{e_i, f_i, h_i, z_i\}_{i=0,1}$ and relations (22), (23) and (25) constitute an infinite-dimensional algebra called super-Yangian of the Lie superalgebra $gl(1|1)$ and denoted by $Y(gl(1|1))$. $Y(gl(1|1))$ is a Hopf algebra with the comultiplication Δ, co-unit ϵ and antipode S defined, respectively, by

$$\begin{aligned}
&\Delta(T(u)_{ab}) = \sum_c T(u)_{ac} \otimes T(u)_{cb} \\
&\epsilon(T(u)) = 1 \\
&S(T(u)) = T(u)^{-1}.
\end{aligned} \tag{26a}$$

If writing the Hopf structure in terms of operators $\{e_i, f_i, h_i, z_i\}_{i=0,1}$, we obtain the following forms:

$$\begin{aligned}
&\Delta(X) = 1 \otimes X + X \otimes 1 \\
&\Delta(e_1) = 1 \otimes e_1 + e_1 \otimes 1 - \frac{C_0}{C_1}(h_0 \otimes e_0 + e_0 \otimes h_0) + \frac{C_0\gamma_0}{2C_1}(z_0 \otimes e_0 - e_0 \otimes z_0) \\
&\Delta(f_1) = 1 \otimes f_1 + f_1 \otimes 1 - \frac{C_0}{C_1}(h_0 \otimes f_0 + f_0 \otimes h_0) + \frac{C_0\gamma_0}{2C_1}(-z_0 \otimes f_0 + f_0 \otimes z_0) \\
&\Delta(z_1) = 1 \otimes z_1 + z_1 \otimes 1 + \frac{C_0}{C_1}(e_0 \otimes f_0 - f_0 \otimes e_0) - \frac{C_0}{C_1}(z_0 \otimes h_0 + h_0 \otimes z_0) \\
&\Delta(h_1) = 1 \otimes h_1 + h_1 \otimes 1 - \frac{C_0\gamma_0}{2C_1}(f_0 \otimes e_0 + e_0 \otimes f_0) - \frac{C_0}{C_1}h_0 \otimes h_0 - \frac{C_0\gamma_0^2}{4C_1}z_0 \otimes z_0 \\
&S(X) = -X \\
&S(e_1) = -e_1 - \frac{C_0}{C_1}(h_0e_0 + e_0h_0) \\
&S(f_1) = -f_1 - \frac{C_0}{C_1}(h_0f_0 + f_0h_0) \\
&S(z_1) = -z_1 - \frac{C_0}{C_1}(f_0e_0 - e_0f_0 + 2z_0h_0) \\
&S(h_1) = -h_1 - \frac{C_0\gamma_0}{2C_1}(f_0e_0 + e_0f_0) - \frac{C_0\gamma_0^2}{4C_1}z_0z_0 - \frac{C_0}{C_1}h_0h_0 \\
&\epsilon(1) = 1 \qquad \epsilon(X) = \epsilon(Y) = 0
\end{aligned} \tag{26b}$$

where $X = e_0, f_0, z_0, h_0, Y = e_1, f_1, z_1, h_1$.

Now we introduce a set of bosonic oscillators $b_i, b_i^\dagger$ and a set of fermionic oscillators $a_i, a_i^\dagger$ satisfying

$$\begin{cases}
\{a_i, a_j^\dagger\} = [b_i, b_j^\dagger] = \delta_{ij} \\
\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0 \\
[a_i, b_j] = [a_i^\dagger, b_j^\dagger] = [a_i^\dagger, b_j] = [a_i, b_j^\dagger] = 0.
\end{cases} \tag{27}$$

Identifying

$$\begin{cases} e_0 = \sum_i b_i^\dagger a_i & f_0 = \sum_i a_i^\dagger b_i \\ z_0 = \sum_i (a_i^\dagger a_i + b_i^\dagger b_i) & h_0 = \sum_i b_i^\dagger b_i \\ e_1 = \sum_{i,j} A_{ij} b_i^\dagger a_j + \sum_{i,j} B_{ij} b_i^\dagger a_i (a_j^\dagger a_j + b_j^\dagger b_j) & \\ f_1 = \sum_{i,j} A_{ij} a_i^\dagger b_j - \sum_{i,j} B_{ij} a_i^\dagger b_i (a_j^\dagger a_j + b_j^\dagger b_j) & \\ z_1 = \sum_{i,j} A_{ij} (a_i^\dagger a_j + b_i^\dagger b_j) & \\ h_1 = \frac{1}{2} \sum_{i,j} A_{ij} (-a_i^\dagger a_j + b_i^\dagger b_j) + \sum_{i,j} B_{ij} b_i^\dagger a_i a_j^\dagger b_j & \end{cases} \tag{28}$$

where A_{ij}, B_{ij} are parameters and $B_{ij}+B_{ji}=0$. We can prove that equations (28) reproduce the commutation relations given in equations (22) and (23). Substituting equations (28) into Serre relations (25), there are some constraints on A_{ij}, B_{ij} and they will be related to parameters C_0, C_1.

In this paper, we only discuss the super-Yangian of the Lie superalgebra $gl(1|1)$ and its oscillator realization. The question we should answer is how to generalize the discussion to the case of superalgebra $gl(m|n)$ and other superalgebras. However, this is connected with physical problems, i.e. whether integrable models exist with $\mathcal{R}$ matrix associated with Lie superalgebras. As a first step, we wish to find a model with the super-Yangian symmetry we have discussed. This problem asks for a further study of super-Yangian and its representation theory.

From the above discussion, we see that (super-)Yangian is related to the (graded) RTT relation. Actually, there are dual relations to the (graded) RTT relation, their corresponding algebras are not contained in the (super-)Yangian. Yangian double considers all algebraic information contained in RTT relation and its dual relations. The Yangian double for simple Lie algebras has recently become an interesting research object [13, 14]. Naturally, the super-Yangian double and the related problems also need to be studied. Work in this respect is under investigation.

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